Kinetic Equations

Solution to the Exercises

– 25.03.2021 –

Teachers: Prof. Chiara Saffirio, Dr. Théophile Dolmaire

Assistant: Dr. Daniele Dimonte – daniele.dimonte@unibas.ch

Exercise 1

Consider a family of probability measures $F \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$. We say that *the family is tight* if for any $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{R}^d$ such that $\mu(K) > 1 - \varepsilon$ for any $\mu \in F$.

The following Theorem holds true (for a proof, see Theorem 5.1 in *Convergence of Probability Measures (Second Edition)* by P. Billingsley).

Theorem (Prohorov's Theorem)**.** *Consider a sequence of probability measures $\{\mu_k\}_{k\in\mathbb{N}} \subseteq \mathcal{P}\left(\mathbb{R}^d\right)$ which is tight. Then there exists a subsequence*[1] *$\{\mu_{k_l}\}_{l\in\mathbb{N}}$ and a probability measure $\mu \in \mathcal{P}\left(\mathbb{R}^d\right)$ such that $\mu_{k_l} \rightharpoonup \mu$ if $l \to +\infty$.*

Use Prohorov's Theorem (without proving it) to prove that $\left(\mathcal{P}_1\left(\mathbb{R}^d\right), \mathcal{W}_1\right)$ is a complete metric space.

Hint: Consider a Cauchy sequence for $\mathcal{W}_1$, show that it is tight. Using the Theorem deduce the existence of a weak limit and prove that the convergence holds also with the metric $\mathcal{W}_1$.

Proof. Let $\{\mu_k\}_{k\in\mathbb{N}} \subseteq \mathcal{P}_1\left(\mathbb{R}^d\right)$ be a Cauchy sequence, i.e., for any ε exists $N \in \mathbb{N}$ such that

$$\mathcal{W}_1(\mu_k, \mu_l) < \varepsilon \qquad\qquad k,\ l \in \mathbb{N},\ k > N,\ l > N. \tag{1}$$

We first show that $\{\mu_k\}_{k\in\mathbb{N}} \subseteq \mathcal{P}_1\left(\mathbb{R}^d\right)$ is tight. Indeed let N such that

$$\mathcal{W}_1(\mu_k, \mu_l) < 1 \qquad\qquad k,\ l \in \mathbb{N},\ k > N,\ l > N. \tag{2}$$

As a consequence we get by the definition of Wasserstein distance that if $k > N$

$$\int_{\mathbb{R}^d} |x|\, d\mu_k(x) = \int_{\mathbb{R}^d} |x|\, d\mu_k(x) - \int_{\mathbb{R}^d} |x|\, d\mu_{N+1}(x) + \int_{\mathbb{R}^d} |x|\, d\mu_{N+1}(x) \tag{3}$$

$$\leqslant \mathcal{W}_1(\mu_k, \mu_{N+1}) + \int_{\mathbb{R}^d} |x|\, d\mu_{N+1}(x) \leqslant 1 + \int_{\mathbb{R}^d} |x|\, d\mu_{N+1}(x), \tag{4}$$

and therefore we get

$$\sup_{k\in\mathbb{N}} \int_{\mathbb{R}^d} |x|\, d\mu_k(x) \leqslant \max\left\{\max_{1\leqslant k\leqslant N} \int_{\mathbb{R}^d} |x|\, d\mu_k(x), 1 + \int_{\mathbb{R}^d} |x|\, d\mu_{N+1}(x)\right\} \tag{5}$$

$$=: C < +\infty. \tag{6}$$

[1] Recall that $\{\mu_{k_l}\}_{l\in\mathbb{N}}$ is a subsequence of $\{\mu_k\}_{k\in\mathbb{N}}$ if the sequence $\{k_l\}_{l\in\mathbb{N}}$ is a sequence of natural numbers such that $k_{l+1} > k_l$ for any $l \in \mathbb{N}$.

We then consider $R > 0$ a positive real number and B_R the closed ball centered in the origin of radius R to get

$$\inf_{k\in\mathbb{N}} \int_{B_R} d\mu_k(x) = \inf_{k\in\mathbb{N}} \left(1 - \int_{B_R^c} d\mu_k(x)\right) = 1 - \sup_{k\in\mathbb{N}} \int_{B_R^c} d\mu_k(x) \tag{7}$$

$$\geqslant 1 - \sup_{k\in\mathbb{N}} \int_{B_R} \frac{|x|}{R} d\mu_k(x) \geqslant 1 - \frac{C}{R} \tag{8}$$

where in the first inequality we used the fact that outside the ball B_R we have $\frac{|x|}{R} \geqslant 1$.

So, fix now $\varepsilon > 0$; we then have $\mu_k\left(B_{\frac{C}{\varepsilon}}\right) \geqslant 1 - \varepsilon$ for any $k \in \mathbb{N}$ and given that the closed ball is compact, the sequence is tight.

Now, Prohorov's Theorem ensures us that there exists a subsequence $\{\mu_{k_l}\}_{l\in\mathbb{N}}$ and a probability measure $\mu \in \mathcal{P}\left(\mathbb{R}^d\right)$ such that $\mu_{k_l} \rightharpoonup \mu$ if $l \to +\infty$. We first show that this implies that $\mathcal{W}_1(\mu_{k_l}, \mu) \to 0$ as $l \to +\infty$.

Recall from Exercise 2 of the sheet from the 18.03.2021 (third sheet) the family of functions $f_\varepsilon \in C_\mathrm{b}\left(\mathrm{R}^d\right)$, defined such that

$$\chi_{B_R}(x) \leqslant f_\varepsilon(x) \leqslant \chi_{B_{R+\varepsilon}}(x). \tag{9}$$

By definition of weak convergence and from the bound in (5) we get that for any $R > 0$

$$\int_{B_R} |x|\, d\mu(x) \leqslant \int_{\mathbb{R}^d} |x|\, f_\varepsilon(x)\, d\mu(x) = \lim_{l\to+\infty} \int_{\mathbb{R}^d} |x|\, f_\varepsilon(x)\, d\mu_{k_l}(x) \tag{10}$$

$$\leqslant \limsup_{l\to+\infty} \int_{\mathbb{R}^d} |x|\, d\mu_{k_l}(x) \leqslant C. \tag{11}$$

Taking the supremum over $R > 0$ this implies that $\mu \in \mathcal{P}_1\left(\mathbb{R}^d\right)$.

Consider now the following function:

$$g_R(x) := \begin{cases} |x|, & |x| \leqslant R, \\ 2R - |x|, & R < |x| \leqslant 2R, \\ 0, & |x| > 2R. \end{cases} \tag{12}$$

This is a Lipschitz function with $|x|\,\chi_{B_R}(x) \leqslant g_R(x) \leqslant |x|$ and $\|g_R\|_{\mathrm{Lip}(\mathbb{R}^d)} \leqslant 1$, therefore we get that for any $l \in \mathbb{N}$

$$\int_{B_R} |x|\, d\mu_{k_l}(x) \leqslant \int_{\mathbb{R}^d} g_R(x)\, d\mu_{k_l}(x) \tag{13}$$

$$= \int_{\mathbb{R}^d} g_R(x)\, d\mu_{k_l}(x) - \int_{\mathbb{R}^d} g_R(x)\, d\mu_{k_n}(x) + \int_{\mathbb{R}^d} g_R(x)\, d\mu_{k_n}(x) \tag{14}$$

$$\leqslant \|g_R\|_{\mathrm{Lip}(\mathbb{R}^d)}\, \mathcal{W}_1(\mu_{k_l}, \mu_{k_n}) + \int_{\mathbb{R}^d} g_R(x)\, d\mu_{k_n}(x) \tag{15}$$

$$\leqslant \mathcal{W}_1(\mu_{k_l}, \mu_{k_n}) + \int_{\mathbb{R}^d} g_R(x)\, d\mu_{k_n}(x). \tag{16}$$

Fix $\varepsilon > 0$; given that $\mu_{n_k} \rightharpoonup \mu$ as $k \to +\infty$ there exists N such that for any $n > N$ we get

$$\left|\int_{\mathbb{R}^d} g_R(x)\, d\mu_{k_n}(x) - \int_{\mathbb{R}^d} g_R(x)\, d\mu(x)\right| < \frac{\varepsilon}{2}. \tag{17}$$

Therefore, using also (5) we get that for any $n > N$

$$\int_{\mathbb{R}^d} |x|\, d\mu_{k_l}(x) \leqslant \int_{B_R} |x|\, d\mu_{k_l}(x) + \frac{C}{R} \tag{18}$$

$$\leqslant \mathcal{W}_1(\mu_{k_l}, \mu_{k_n}) + \int_{\mathbb{R}^d} g_R(x)\, d\mu_{k_n}(x) + \frac{C}{R} \tag{19}$$

$$\leqslant \mathcal{W}_1(\mu_{k_l}, \mu_{k_n}) + \int_{\mathbb{R}^d} g_R(x)\, d\mu(x) + \frac{\varepsilon}{2} + \frac{C}{R} \tag{20}$$

$$\leqslant \mathcal{W}_1(\mu_{k_l}, \mu_{k_n}) + \int_{\mathbb{R}^d} |x|\, d\mu(x) + \frac{\varepsilon}{2} + \frac{C}{R}. \tag{21}$$

Using the fact that the sequence is Cauchy we also get that there is $N' \geqslant N$ such that for any $l > N'$ we get

$$\int_{\mathbb{R}^d} |x|\, d\mu_{k_l}(x) \leqslant \int_{\mathbb{R}^d} |x|\, d\mu(x) + \varepsilon + \frac{C}{R}. \tag{22}$$

This implies in particular that

$$\limsup_{l\to+\infty} \int_{\mathbb{R}^d} |x|\, d\mu_{k_l}(x) \leqslant \int_{\mathbb{R}^d} |x|\, d\mu(x). \tag{23}$$

From Exercise 2 of the sheet from the 18.03.2021 (third sheet), this together with the weak convergence implies that for any function $\varphi \in C(\mathbb{R}^d)$ such that $|\varphi(x)| \leqslant C(1+|x|)$ we get

$$\lim_{l\to+\infty} \int_{\mathbb{R}^d} \varphi(x)\, d\mu_{k_l}(x) = \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x). \tag{24}$$

This in particular implies the statement for Lipschitz functions (given that $\mathrm{Lip}(\mathbb{R}^d) \subseteq C(\mathbb{R}^d)$ and $|\varphi(x)| \leqslant |\varphi(0)| + \|\varphi\|_{\mathrm{Lip}(\mathbb{R}^d)} |x|$).

Similarly as before, from the definition of $\mathcal{W}_1$ we now get

$$\int_{\mathbb{R}^d} \varphi(x)\, d\mu_{k_l}(x) - \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x) \leqslant \tag{25}$$

$$\leqslant \mathcal{W}_1\left(\mu_{k_l}, \mu_{k_{l+n}}\right) + \left|\int_{\mathbb{R}^d} \varphi(x)\, d\mu_{k_{l+n}}(x) - \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x)\right|. \tag{26}$$

Fix $\varepsilon > 0$; there exists $N \in \mathbb{N}$ such that for any $l > N$ we get $\mathcal{W}_1\left(\mu_{k_l}, \mu_{k_{l+n}}\right) < \varepsilon$. Given that we have that the left hand side does not depend on n, we get

$$\int_{\mathbb{R}^d} \varphi(x)\, d\mu_{k_l}(x) - \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x) \leqslant \tag{27}$$

$$\leqslant \varepsilon + \inf_{n\in\mathbb{N}} \left|\int_{\mathbb{R}^d} \varphi(x)\, d\mu_{k_{l+n}}(x) - \int_{\mathbb{R}^d} \varphi(x)\, d\mu(x)\right| = \varepsilon. \tag{28}$$

Given that N does not depend on φ this implies $\mathcal{W}_1(\mu_{k_l},\mu)\to 0$ as $l\to+\infty$.

We finally show that this implies that the full sequence converges. To prove that, consider by contradiction that there exists a subsequence $\{\mu_{k_l}\}_{l\in\mathbb{N}}$ such that

$$\liminf_{l\to+\infty}\mathcal{W}_1(\mu_{k_l},\mu)=C>0. \tag{29}$$

We can then construct a sub-subsequence $\left\{\mu_{k_{l_n}}\right\}_{n\in\mathbb{N}}$ such that $\mathcal{W}_1\left(\mu_{k_{l_n}},\mu\right)\to 0$ as $n\to+\infty$, which gives a contradiction.

□

Exercise 2

Prove the second part of Dobrushin's Theorem, i.e., let $M=\{\mu_t|\ t\in[0,T]\}\in\mathfrak{M}_T^+(\mu_0)$ a solution to

$$\begin{cases}\partial_t\mu_t(\psi)=\mu_t(v\cdot\nabla_x\psi+E_{\mu_t}\cdot\nabla_v\psi), & t\in[0,T],\ \psi\in C_c^\infty\left(\mathbb{R}^3\times\mathbb{R}^3\right),\\ \mu_t(\psi)|_{t=0}=\mu_0(\psi), & \psi\in C_c^\infty\left(\mathbb{R}^3\times\mathbb{R}^3\right),\end{cases} \tag{30}$$

with

$$E_\mu(x)=\int_{\mathbb{R}^3\times\mathbb{R}^3}\nabla U(x-x')\,d\mu(x',v'),\qquad U\in C_b^2\left(\mathbb{R}^3\right), \tag{31}$$

and with μ_0 absolutely continuous with respect to $\mathcal{L}^6$, i.e. $d\mu_0(x,v)=f_0(x,v)\,dxdv$. Prove that if $f_0\in C^1\left(\mathbb{R}^3\times\mathbb{R}^3\right)$ then also μ_t is absolutely continuous with respect to $\mathcal{L}^6$ and moreover $d\mu_t(x,v)=f(t,x,v)\,dxdv$ with $f\in C^1\left([0,T]\times\mathbb{R}^3\times\mathbb{R}^3\right)$.

Hint: It can be convenient to use the fact that, once the solution exists, the Vlasov equation can be seen as a Liouville equation with a potential depending on the existing solution.

Proof. To get the result, first notice that the measure μ_t is solution to the problem

$$\begin{cases}\partial_t\mu_t(\psi)=\mu_t(b(t,\cdot)\cdot\nabla\psi), & t\in[0,T],\ \psi\in C_c^\infty\left(\mathbb{R}^3\times\mathbb{R}^3\right),\\ \mu_t(\psi)|_{t=0}=\mu_0(\psi), & \psi\in C_c^\infty\left(\mathbb{R}^3\times\mathbb{R}^3\right),\end{cases} \tag{32}$$

with $b(t,x,v)=(v,E_{\mu_t}(t,x))$. If we then prove that $b\in C^1\left([0,T]\times\mathbb{R}^6;\mathbb{R}^6\right)$ is bounded with $\nabla_z b\in L^\infty\left([0,T]\times\mathbb{R}^6;M_6(\mathbb{R})\right)$, we can then apply Exercise 1 of the sheet from the 18.03.2021 (third sheet) to get the result. This follows easily from the definition of E_μ.

□

Exercise 3

Let $f_0\in L^1\cap L^\infty\left(\mathbb{R}^3\times\mathbb{R}^3\right)$; consider the following initial value problem:

$$\begin{cases}\partial_t f+v\cdot\nabla_x f=0, & \text{in }\mathcal{D}'\left([0,+\infty)\times\mathbb{R}^3\times\mathbb{R}^3\right),\\ f|_{t=0}=f_0, & \text{in }\mathcal{D}'\left(\mathbb{R}^3\times\mathbb{R}^3\right),\end{cases} \tag{33}$$

where we also assume as usual that the map $t\mapsto\langle f(t,\cdot,\cdot),\varphi\rangle$ is continous in t for any $\varphi\in C_c^\infty\left(\mathbb{R}^3\times\mathbb{R}^3\right)$.

(i) Prove that there exists a unique solution to (33) and show its explicit form.

(ii) Use the explicit form to prove that

$$\|f(t,\cdot,\cdot)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} = \|f_0\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}, \qquad \forall t\in[0,+\infty),\ p\in[1,+\infty]. \tag{34}$$

(iii) Use the explicit form to prove the following dispersion relation:

$$\|f(t,\cdot,\cdot)\|_{L^\infty_x(\mathbb{R}^3;L^1_v(\mathbb{R}^3))} \leqslant \frac{1}{|t|^3}\|f_0\|_{L^1_x(\mathbb{R}^3;L^\infty_v(\mathbb{R}^3))}, \qquad \forall t\in(0,+\infty). \tag{35}$$

Proof. To prove (i), is enough to use the explicit solution. Indeed, we know that for any $T>0$, $t\in[-T,T]$ the function $f(t,(x,v)) = f_0(Z(0,t,(x,v)))$ is a solution to the problem in $[-T,T]$, where $Z(s,t,(x,v)) = (X(s,t,(x,v)), V(s,t,(x,v)))$ solves

$$\begin{cases} \partial_s X(s,t,(x,v)) = V(s,t,(x,v)), & \forall (s,t,x,v)\in[-T,T]\times[-T,T]\times\mathbb{R}^3\times\mathbb{R}^3, \\ \partial_x V(s,t,(x,v)) = 0, & \forall (s,t,x,v)\in[-T,T]\times[-T,T]\times\mathbb{R}^3\times\mathbb{R}^3, \\ Z(t,t,(x,v)) = (x,v), & \forall (t,x,v)\in[-T,T]\times\mathbb{R}^3\times\mathbb{R}^3, \end{cases} \tag{36}$$

Given that $Z(s,t,(x,v)) = (x-tv,v)$, it is then easy to prove that $f(t,(x,v)) = f_0(x-tv,v)$ is the unique solution in $[-T,T]$, and it can naturally extended to a solution on $\mathbb{R}$. Finally, uniqueness on $\mathbb{R}$ follows from uniqueness on every intervall $[-T,T]$.

To prove (ii) we have that for any $p\in[1,+\infty)$

$$\|f(t,\cdot,\cdot)\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f(t,(x,v))|^p\,dxdv\right)^{\frac{1}{p}} \tag{37}$$

$$= \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f_0(x-tv,v)|^p\,dxdv\right)^{\frac{1}{p}} \tag{38}$$

$$= \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |f_0(x,v)|^p\,dxdv\right)^{\frac{1}{p}} = \|f_0\|_{L^p(\mathbb{R}^3\times\mathbb{R}^3)}. \tag{39}$$

Similarly, we also get

$$\|f(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)} = \operatorname*{ess\,sup}_{(x,v)\in\mathbb{R}^d\times\mathbb{R}^d} |f(t,(x,v))| = \operatorname*{ess\,sup}_{(x,v)\in\mathbb{R}^d\times\mathbb{R}^d} |f_0(x-tv,v)| \tag{40}$$

$$= \operatorname*{ess\,sup}_{(x,v)\in\mathbb{R}^d\times\mathbb{R}^d} |f_0(x,v)| = \|f_0\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}. \tag{41}$$

To prove (iii) we have

$$\|f(t,\cdot,\cdot)\|_{L^\infty_x(\mathbb{R}^3;L^1_v(\mathbb{R}^3))} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|f(t,(x,v))|\,dv = \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}|f_0(x-tv,v)|\,dv \tag{42}$$

$$= \operatorname*{ess\,sup}_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}\left|f_0\left(x',\frac{x-x'}{t}\right)\right|\frac{1}{|t|^3}dx' \tag{43}$$

$$\leqslant \frac{1}{|t|^3}\int_{\mathbb{R}^d}\operatorname*{ess\,sup}_{v\in\mathbb{R}^d}|f_0(x,v)|\,dx = \frac{1}{|t|^3}\|f_0\|_{L^1_x(\mathbb{R}^3;L^\infty_v(\mathbb{R}^3))}. \tag{44}$$

□